Conservative Interpolation of Vector Fields

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New Zealand eScience Infrastructure
Overview

• What makes interpolation conservative for vector fields
• Why we need conservative interpolation
• Some results and some new applications

This work is supported by the next generation climate/weather model development effort, LFRic, led by UK Met Office with contributions from international partners (incl. NIWA)

LFRic’s aim is to develop new infrastructure to allow weather/climate models to scale on current and next generation computers

A significant change from the current weather/climate operational model (Unified Model or UM) is the introduction of the cubed-sphere grid

LFRic’s discretisation is mimetic – a game changer

- Mixed finite element discretisation satisfies:
  - $\text{curl } \text{grad} = 0$
  - $\text{div } \text{curl} = 0$
- Need interpolation that preserves above properties for
  - Regridding
  - Visualisation
  - Advection
  - ?

Face and edge centred fields are also known as Arakawa C/D grid staggering

4 types of field - 4 types of staggerings

“New” kids on the block
Vector fields need this

Order 1
- $Q_1$
- $N_{c_1}^e$
- $N_{c_1}^f$
- $dQ_0$

Order 2
- $Q_2$
- $N_{c_2}^e$
- $N_{c_2}^f$
- $dQ_1$

Periodic table of finite elements http://femtable.org
What makes interpolation mimetic/conservative

- **Integrals are invariant** after regridding
- Depending on the type of field, the integrals can be over:
  - volumes (e.g. **total mass** is conserved when integrating density)
  - surfaces (e.g. conservation of **total magnetic flux**)
  - lines (e.g. conservation of **vorticity** when integrating velocity)

\[
\begin{align*}
\text{Edge centred} & \quad \int \alpha \cdot dx \\
\text{Face centred} & \quad \int \alpha \cdot dS \\
\text{Cell centred} & \quad \int \alpha dV \\
\hline
\end{align*}
\]
Work with integrals

- Interpolation returns a scalar representing the total mass/flux/voltage over a target volume/surface/line.

\[ \int f = \sum_i f_i \int \phi_i = \sum_i f_i w_i \]

- Basis function integrated over the target.
- Interpolated field is integrated over target volume/surface/line (scalar).
- Interpolation weight (scalar).
- Integrated field values over cells/faces/edges (scalar).

All quantities are scalars.

No need to worry about coordinate transformations!

Result 1: field with a polar singularity

- Exact flux/line integral when the contour does not cut through the singular point

![Graph showing a field with a polar singularity and residual error analysis.](Image)

- Large bilinear error when contour passes close to singularity
- Mimetic is exact
- Contour intersects cell containing singularity
- Residual error of bilinear interp.
Results 2: compute streamlines

- Not using the correct basis functions causes **red** streamlines to spiral
- **Green** are the mimetic streamlines
- **Blue** are the exact streamlines

Works at corner of cubed sphere (no visible artefacts)
Results 3: mimetic interpolation conserves many orders of magnitude better than bilinear

- Regridding on the cubed-sphere grid

Bilinear does not conserve

Conservation error

Conservation error accumulates with higher resolution

$10^{12}$
How regridding can be used to advect a vector field (work in progress)

- Advection of 1- or 2-form
- Discretisation of Lie operator
- Use sweep/extrusion method
  - Integrate 1-form over line (S) and time
  - Time extrudes the line integral which becomes an area integral

\[
\frac{\partial \alpha}{\partial t} + \mathcal{L}_u \alpha
\]

\[
\mathcal{L}_u \equiv \text{integrate 1-form over line (S) and time}
\]

\[
\int dt \int_S i_u d\alpha = \oint_{\partial (E \otimes S)} \alpha
\]

\[
\int dt \int_S d(i_u \alpha) = \int_{E \otimes \partial S} \alpha
\]

Upstream advection for 1- or 2-forms (a.k.a. vectors)

\[ \int_{S(t)} \alpha(t) - \int_{S(t)} \alpha(0) + \int_{S(t)} \alpha(0) - \int_{S(0)} \alpha(0) = 0 \]

- Step 1: integrate grid back in time ("upstream")
  \[ \int_{S(t)} \alpha(t) = \int_{S(0)} \alpha(0) \]

- Step 2: regrid the old field on the "upstream" grid

Again conservation!

Black: grid a new time "t"; Orange is grid at time 0

Lie derivative term
Conclusion

• Think in terms of integrals
• Choice of basis functions matters
• Mimetic interpolation:
  • Is invariant to coordinate transformation
  • Handles singularities and masking out-of-the-box
  • Can compute fluxes, advect fields...

• Are you mimetic?

Staggering  | Nodal | Edge | Face | Cell
---|---|---|---|---
Target      | Point | Line integral | Flux integral | Volume integral
Method      | Linear interp “Babylonian” | “NEW” | “NEW” | Conservative interp
            | | | | Jones 1996

Whitney forms 1950s

Thank you